JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **32**, No. 4, November 2019
http://dx.doi.org/10.14403/jcms.2019.32.4.529

EXPANSIVENESS ON INVERTIBLE ITERATED FUNCTION SYSTEMS

Hahng-Yun Chu, Minhee Gu, and Se-Hyun Ku∗

Abstract. In this article, we study the dynamics of invertible iterated function systems on compact spaces. We prove the equivalences for the notions of expansiveness on invertible iterated function systems.

1. Introduction

Let (X, d) be a compact metric space and Λ a nonempty finite set. Let us recall that an *Iterated Function System*(shortly, IFS) $\mathcal{F} := \{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ is a family of continuous maps $f_\lambda : X \to X$, $\lambda \in \Lambda$. We denote a typical element of $\Lambda^{\mathbb{N}}$ as $\sigma = \{\lambda_1, \lambda_2, \cdots\}$. An IFS has an important role in the field of fractal dynamical systems and it is used for the construction of deterministic fractals and can be found in various applications like image compression and image processing. The notion of IFS was introduced in order to unify the way to generate a broad class of fractals and the study of IFSs helps to understand dynamical systems. The current form of IFSs was introduced by Hutchinson [7]. Barnsley and Vince [2] showed that the Conley's attractor can be extended to IFS. Nia [8, 9] investigated dynamic properties in IFS and specially dealt with the various types of shadowing properties and their equivalences in IFSs. In recent years, there are a lot of results about dynamics on the systems(see [5, 6]).

A homeomorphism f on a metric space (X, d) is *expansive* if there exists a positive real number e such that $x, y \in X$ with $x \neq y$ implies

Received November 11, 2019; Accepted November 14, 2019.

2010 Mathematics Subject Classification: Primary 54H20, Secondary 37B25.

Key words and phrases: iterated function systems, invertible iterated function systems, expansive, generators.

This work was financially supported by research fund of Chungnam National University in 2016.

∗ Corresponding author.

$d(f^n x, f^n y) > e$ for some integer n. This notion was first introduced by Utz [10]. He proved properties about the notion related with asymptotic trajectories and the powers of expansive homeomorphisms. Bryant [3, 4] investigated the results concerning expansive homeomorphisms to compact uniform spaces and also proved that topological conjugate of an expansive homeomorphism is also an expansive homeomorphism. Recently, many authors studied a lot of modified notions of expansiveness and their relations on IFSs.

In this paper, we study that the notion of expansiveness in dynamical systems can be extended to invertible iterated function systems (shortly, IIFSs) and investigate the properties of expansiveness in IIFSs. We also introduce the notions of generators in IIFSs. In the systems, we consider two types of generators, in detail, generators and rigid generators. We show that for IIFS on compact metric spaces, the existences of the generators and rigid generators in IIFS are equivalent to the notions of expansiveness and rigid expansiveness on the systems, respectively.

From now on, let (X, d) be a compact metric space and Λ a nonempty finite set.

2. Expansiveness and rigid expansiveness on IIFS

Let $\mathcal{F} = \{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ be an IFS on a compact metric space (X, d). The IFS $\mathcal{F}$ is *expansive* if there exists a positive constant e such that $x \neq y$ $(x, y \in X)$ implies

$$d(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(x), f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(y)) > e$$

for some $n \in \mathbb{N}$ and $\lambda_i \in \Lambda$, $i = 1, \cdots, n$. Here, e is called an *expansive constant* for $\mathcal{F}$. For $k \geq 1$, put

$$\mathcal{F}^k = \{f_{\lambda_k} \circ \cdots \circ f_{\lambda_1} : X \to X \mid \lambda_1, \cdots, \lambda_k \in \Lambda\}.$$

Now we deal with IIFSs and notions of expansiveness on the systems which are induced from the notion of expansiveness on IFSs. Let (X, d) be a compact metric space and $\mathcal{F}^+ = \{f_\lambda : X \to X \mid \lambda \in \Lambda^+\}$ an IFS. If every continuous mapping f_λ in $\mathcal{F}^+$ is a homeomorphism from X to itself, we define a new index λ^{-1} satisfying the property that $f_\lambda \circ f_{\lambda^{-1}} = f_{\lambda^{-1}} \circ f_\lambda = id_X$, that is, $(f_\lambda)^{-1} = f_{\lambda^{-1}}$. The extension of index set to define an IIFS is possible when f_λ is a homeomorphism for every $\lambda \in \Lambda^+$.

Let $\mathcal{F}^+ = \{f_\lambda : X \to X \mid \lambda \in \Lambda^+\}$ be an IFS consists of just homeomorphisms from X to itself such that $f_{\lambda_1} \circ f_{\lambda_2} \neq id_X$ for all λ_1 and $\lambda_2 \in$

Λ^+. We define an *invertible IFS* (in short, IIFS) $\mathcal{F}$ from the IFS $\mathcal{F}^+$ given by $\mathcal{F} := \{f_\lambda,\ f_{\lambda^{-1}} \mid \lambda \in \Lambda^+\}$. We denote $\Lambda^- := \{\lambda^{-1} | \lambda \in \Lambda^+\}$ and $\Lambda := \Lambda^+ \cup \Lambda^-$. Then we get that $\mathcal{F} = \{f_\lambda \mid \lambda \in \Lambda\}$.

An IIFS $\mathcal{F}$ is *expansive* if there exists a positive constant e such that for each $x, y \in X$ with $x \neq y$, there exist a positive integer n and $(\lambda_1, \cdots, \lambda_n) \in \Lambda^n$ satisfying

$$d(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(x), f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(y)) > e.$$

Here, e is called an *expansive constant* for $\mathcal{F}$.

An IIFS $\mathcal{F}$ is *rigidly expansive* provided that if there exists a sequence $\sigma = \{\lambda_i\}_{i=1}^\infty \in \Lambda^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$,

$$d(f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(x), f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(y)) \leq e$$

then $x = y$. Here, e is called a *rigidly expansive constant* for $\mathcal{F}$.

Remark 2.1. If an expansive IFS satisfies the commutative property, then the IIFS generated by the IFS is expansive. That is, if an IFS $\mathcal{F}^+ = \{f_\lambda : X \to X \mid \lambda \in \Lambda^+\}$ satisfies that

$$f_{\lambda_1} \circ f_{\lambda_2} = f_{\lambda_2} \circ f_{\lambda_1} \text{ for all } \lambda_1 \text{ and } \lambda_2 \in \Lambda^+,$$

then the IIFS $\mathcal{F}$ is expansive.

Let $\sigma = \{\lambda_i\}_{i=1}^\infty \in \Lambda^{\mathbb{N}}$. We denote the compositions of f_{λ_i} by

$$\mathcal{F}_{\Sigma(\sigma)_n}(x) := \begin{cases} f_{\lambda_n} \circ \cdots \circ f_{\lambda_1}(x) & \text{for } n > 0, \\ x & \text{for } n = 0. \end{cases}$$

And $\mathcal{F}^{-1}_{\Sigma(\sigma)_n}$ denotes the inverse of $\mathcal{F}_{\Sigma(\sigma)_n}$.

Let f be a homeomorphism of a compact metric space X into itself. It is well known fact that f is expansive if and only if (X, f) has a generator. It appears likely that this will be a very useful technique in studyng expansive homeomorphisms. From now on $\mathcal{N} = \{0, 1, 2, \cdots\}$ denotes the nonnegative integers.

Definition 2.2. Let (X, d) be a compact metric space and α be a finite open cover of X. Let $\mathcal{F} = \{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ be an IIFS on which $f_\lambda : X \to X$ is a homeomorphism for every $\lambda \in \Lambda$.

The open cover α is a *generator* for $\mathcal{F}$ if for every sequence $\{A_n\}_{n \in \mathcal{N}}$ in α,

$$\bigcap_{\sigma \in \Lambda^{\mathbb{N}}} \bigcap_{n=0}^{\infty} \mathcal{F}^{-1}_{\Sigma(\sigma)_n}(cl(A_n))$$

is at most one point.

The open cover α is a *weak generator* for $\mathcal{F}$ if for every sequence $\{A_n\}_{n\in\mathcal{N}}$ in α,

$$\bigcap_{\sigma\in\Lambda^{\mathbb{N}}}\bigcap_{n=0}^{\infty}\mathcal{F}^{-1}_{\Sigma(\sigma)_n}(A_n)$$

is at most one point.

The open cover α is a *rigid generator* for $\mathcal{F}$ if for every sequence $\{A_n\}_{n\in\mathcal{N}}$ in α and $\sigma\in\Lambda^{\mathbb{N}}$,

$$\bigcap_{n=0}^{\infty}\mathcal{F}^{-1}_{\Sigma(\sigma)_n}(cl(A_n))$$

is at most one point.

The open cover α is a *weak rigid generator* for $\mathcal{F}$ if for every sequence $\{A_n\}_{n\in\mathcal{N}}$ in α and $\sigma\in\Lambda^{\mathbb{N}}$,

$$\bigcap_{n=0}^{\infty}\mathcal{F}^{-1}_{\Sigma(\sigma)_n}(A_n)$$

is at most one point.

THEOREM 2.3. *Let (X,d) be a compact metric space and let $\mathcal{F}=\{f_\lambda : X\to X \mid \lambda\in\Lambda\}$ be an IIFS. Then the following statements are equvialent.*

(1) *$\mathcal{F}$ is expansive.*
(2) *$\mathcal{F}$ has a generator.*
(3) *$\mathcal{F}$ has a weak generator.*

Proof. It is clear that (2) implies to (3). Now we first prove that (3) implies to (2).

Let (X,d) be a compact metric space and Λ be a nonempty finite set. Let $\mathcal{F}=\{f_\lambda : X\to X \mid \lambda\in\Lambda\}$ be an IIFS which $f_\lambda : X\to X$ are homeomorphisms for every $\lambda\in\Lambda$. Suppose that the IIFS $\mathcal{F}$ has a weak generator $\beta=\{B_1,...,B_s\}$. Let $\delta>0$ be a Lebesgue number for β. Let α be a finite open cover consisting of sets with diameter less than or equal to $\frac{\delta}{2}$. Let $\{A_n\}_{n\in\mathcal{N}}$ be a sequence in α. For every $n\in\mathcal{N}$, there is a positive integer $j_n\in\{1,\cdots,s\}$ such that $cl(A_n)\subseteq B_{j_n}$. So we get that

$$\bigcap_{n=0}^{\infty}\mathcal{F}^{-1}_{\Sigma(\sigma)_n}(cl(A_n))\subseteq\bigcap_{n=0}^{\infty}\mathcal{F}^{-1}_{\Sigma(\sigma)_n}(B_{j_n})$$

which has at most one point. It follows that $\mathcal{F}$ has a generator.

Now we prove the equivalence between (1) and (3). Suppose that α is a weak generator for $\mathcal{F}$. Let $\delta > 0$ be a Lebesgue number for α. To show that δ is an expansive constant for $\mathcal{F}$, we assume that for every $\sigma \in \Lambda^{\mathbb{N}}, d\big(\mathcal{F}_{\Sigma(\sigma)_n}(x),\ \mathcal{F}_{\Sigma(\sigma)_n}(y)\big) \leq \delta$ for all $n \in \mathcal{N}$. Then for each $n \in \mathcal{N}$, there exists $A_n \in \alpha$ such that $\mathcal{F}_{\Sigma(\sigma)_n}(x), \mathcal{F}_{\Sigma(\sigma)_n}(y) \in A_n$. Thus x and y are elements of the set $\bigcap_{n=0}^{\infty} \mathcal{F}^{-1}_{\Sigma(\sigma)_n}(A_n)$ which is at most one point. Thus $x = y$.

Conversely, let $\delta > 0$ be an expansive constant for $\mathcal{F}$ and let α be a finite cover consisting of open balls of radius $\delta/2$. Suppose that x and y are elements of $\bigcap_{\sigma\in\Lambda^{\mathbb{N}}}(\bigcap_{n=0}^{\infty} \mathcal{F}^{-1}_{\Sigma(\sigma)_n}(A_n))$ where $A_n \in \alpha$. Then for every $\sigma \in \Lambda^{\mathbb{N}}$, we obtain that $d\big(\mathcal{F}_{\Sigma(\sigma)_n}(x), \mathcal{F}_{\Sigma(\sigma)_n}(y)\big) \leq \delta$ for all $n \in \mathcal{N}$. Thus $x = y$ by the assumption. □

By an application of the proof of the theorem 2.3, we get the next corollary.

Corollary 2.4. *Let (X, d) be a compact metric space and let $\mathcal{F} = \{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ be an IIFS. Then the following statements are equvialent.*

(1) *$\mathcal{F}$ is rigidly expansive.*
(2) *$\mathcal{F}$ has a rigid generator.*
(3) *$\mathcal{F}$ has a weak rigid generator.*

Let α and β be open covers for X. Define a set $\alpha \vee \beta$ given by

$$\alpha \vee \beta := \{A \cap B \mid A \in \alpha,\ B \in \beta\}.$$

Note that if $f : X \to X$ is a continuous surjection then $f^{-1}(\alpha) = \{f^{-1}(A) \mid A \in \alpha\}$ is also an open cover for X. Let $\mathcal{F} = \{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ be an IIFS on X. For $k \in \mathbb{N}$, we define the family $\mathcal{F}^k$ given by $\mathcal{F}^k = \{f_{\lambda_1} \circ \cdots \circ f_{\lambda_k} : X \to X \mid \lambda_1, \cdots, \lambda_k \in \Lambda\}$.

Theorem 2.5. *Let $\mathcal{F} = \{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ be an IIFS on a compact metric space X and let $k \geq 2$. Then $\mathcal{F}$ has a generator if and only if so does $\mathcal{F}^k$.*

Proof. It is directly obtained that if α is a generator for $\mathcal{F}^k$ then α is also a generator for $\mathcal{F}$.

Conversely, we first assume that α is a generator for $\mathcal{F}$. Define an open cover β of X given by

$$\beta := \alpha \vee \Big(\bigcap_{\lambda_1 \in \Lambda} f_{\lambda_1}^{-1}(\alpha)\Big) \vee \Big(\bigcap_{\lambda_1, \lambda_2 \in \Lambda} (f_{\lambda_1}^{-1} \circ f_{\lambda_2}^{-1})(\alpha)\Big) \vee \cdots$$
$$\cdots \vee \Big(\bigcap_{\lambda_1, \cdots, \lambda_k \in \Lambda} (f_{\lambda_1}^{-1} \circ \cdots \circ f_{\lambda_k}^{-1})(\alpha)\Big).$$

Then we get that β is a generator for $\mathcal{F}^k$. This completes the proof. □

Using the proof of the theorem 2.5, we immediately get that the existences of the notions of weak generator is equivalent between iterations of the IIFS. For rigid generators, we also obtain the following corollary as an application of the above theorem.

Corollary 2.6. *Let $\mathcal{F} = \{f_\lambda : X \to X \mid \lambda \in \Lambda\}$ be an IIFS on a compact metric space X and let $k \geq 2$. Then $\mathcal{F}$ has a rigid generator (weak rigid generator) if and only if so does $\mathcal{F}^k$.*

References

[1] N. Aoki and K. Hiraide, *Topological theory of dynamical systems,* North-Holland, 1994.

[2] M. F. Barnsley and A. Vince, *The Conley attractor of an iterated function system*, Bull. Aust. Math. Soc., **88** (2013), no. 2, 267–279.

[3] B. F. Bryant, *On expansive homeomorphisms*, Pacific J. Math., **10** (1960), 1163–1167.

[4] B. F. Bryant, *Expansive self-homeomorphisms of a compact metric space*, Amer. Math. Monthly, **69** (1962), no. 5, 386–391.

[5] H.-Y. Chu, M. Gu, and S.-H. Ku, *Dynamical properties on iterated function systems,* in preperation

[6] H.-Y. Chu, M. Gu, and S.-H. Ku, *A note on measure expansive iterated function systems,* in preperation

[7] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J., **30** (1981), no. 5, 713–747.

[8] M. F. Nia, *Iterated function systems with the average shadowing property*, Topology Proc., **48** (2016), 261–275.

[9] M. F. Nia and F. Rezaei, *Hartman-Grobman theorem for iterated function systems*, Rocky Mountain J. Math., **49** (2019), 307-333

[10] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc., **1** (1950), 769–774.

Department of Mathematics,
Chungnam National University,
99 Daehak-ro, Yuseong-gu, Daejeon 34134,
Republic of Korea
E-mail: hychu@cnu.ac.kr

Department of Mathematics,
Chungnam National University,
99 Daehak-ro, Yuseong-gu, Daejeon 34134,
Republic of Korea
E-mail: minheegu@cnu.ac.kr

*
Department of Mathematics,
Chungnam National University,
99 Daehak-ro, Yuseong-gu, Daejeon 34134,
Republic of Korea
E-mail: shku@cnu.ac.kr